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ANOMALOUS REFLECTION OF NON-CLASSICAL WAVES FROM ULTRATHIN OBSTACLES†

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The reflection of waves with non-removable discontinuities from ultrathin films is investigated analytically using the example of a hydroelastic one-dimensional linear hyperbolic mixed boundary-value problem of the third type. The wave solution is represented in the form of the sum of the expected and an anomalous component, and a convenient recurrence formula is obtained for calculating the wave patterns after any number of reflections from the film. © 2001 Elsevier Science Ltd. All rights reserved.

When considering contact wave problems of hydroelasticity or of the theory of elasticity with thin obstacles (layers), wave phenomena along the thickness of the obstacle (a shell) are usually neglected and the process is described using a simplified hyperbolic mixed boundary-value problem of the third type [1–3]. In the case of solutions with non-removable discontinuties these problems may exhibit properties not characteristic for classical solutions or solutions with removable discontinuities. One such feature will be considered in more detail below. When describing the material we will use traditional terminology [4]: by classical wave solutions we mean solutions belonging to the class C_2. Solutions with non-removable discontinuities are continuous but have discontinuities of the first derivatives. Solutions with removable discontinuities belong to class C_1, but have discontinuities of the second derivatives.

Consider a linear one-dimensional boundary-value problem of hydroelasticity. We have an infinite plane layer of elastic ideal liquid of thickness L, and the x axis is directed along the normal to the surface of the layer in such a way that the $x = 0$ and $x = L$ planes correspond to the layer surfaces. A specified pressure pulse $f(t)$ is incident on the surface $x = 0$, where f means a pressure perturbation, i.e. the difference between the total and atmospheric pressures at this boundary. The surface $x = L$ is in contact without separation and without leakage, with a thin infinite plate (film) or thickness h and density ρ_p. At the initial instant of time $t = 0$ the liquid is at rest and is under atmospheric pressure P_0. We will neglect the gravity force.

Using the velocity potential of the liquid ψ, we will formulate, following the well-known approaches [1, 3], the corresponding wave mixed boundary-value problem of the third type in the one-dimensional linear formulation

$$
\begin{aligned}
&\psi_{xx} = a^{-2}\psi_{tt}, \quad 0 < x < L, \quad 0 < t \\
&\psi = -\rho_0^{-1}\int_0^t f(t')dt', \quad x = 0, \quad 0 \leqslant t \\
&\psi_x + b\psi = 0, \quad x = L, \quad 0 \leqslant t, \quad b = \rho_0/\rho_p h \\
&\psi = \psi_t = 0, \quad 0 \leqslant x \leqslant L, \quad t = 0
\end{aligned}
\qquad (1)
$$

where ρ_0 is the density of the liquid and a is the velocity of sound in it.

Hence, $b = 0$ corresponds to an absolutely rigid fixed wall $x = L$ and $b = \infty$ corresponds to the free surface. The velocity of the liquid and the pressure perturbation in it are given by, respectively,

$$
v = \psi_x, \quad P = -\rho_0\psi_t \qquad (2)
$$

The problem of a piston, and also some one-dimensional wave problems of the theory of elasticity can be reduced to a boundary-value problem of the type (1).

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We will assume that the external pressure perturbation pulse $f(t)$ satisfies the following requirements.
1. It has finite duration $\tau < L/a$ and is identically equal to zero when $t > \tau$ or $t > 0$.
2. In the interval $[0, \tau]$ it can have a finite number of discontinuities of the first kind.
3. The interval $[0, \tau]$ can be divided into a finite number of intervals, inside which the function $f(t) \in C_1$ and has a bounded derivative $|df/dt| < M$.

With these limitations we can consider the problem using the theory of wave solutions with non-removable discontinuities [4, 5]. We will solve boundary-value problem (1) using d'Alembert's method [2, 6]

$$\psi(x, t) = \theta_1(x - at) + \theta_2(x + at) \tag{3}$$

an be shown that

$$\begin{aligned} \rho_0 a\theta_1(y) &= \int_0^y f\left(-\frac{y'}{a}\right)dy', \quad -2L \leqslant y \leqslant L \\ \rho_0 a\theta_2(y) &= \int_0^{y-2L} f\left(\frac{y'}{a}\right)dy' - 2e^{-b(y-2L)}\int_0^{y-2L} f\left(\frac{y'}{a}\right)e^{by'}dy', \quad 0 \leqslant y \leqslant 3L \end{aligned} \tag{4}$$

Formulae (4) enable us to determine solution (3) in the rectangle $(0 \leqslant x \leqslant L, 0 \leqslant at \leqslant 2L)$. To extend the solution to a wider time interval one needs to extend θ_1 towards negative arguments and θ_2 towards positive arguments. This can be achieved using a recurrence procedure [6], which takes into account the boundary conditions of problem (1).

For the limitations imposed on $f(t)$, functions (4), and of course, solution (3) also, will be continuous everywhere for any value of the arguments. The derivatives of ψ_x and ψ_t in this case can have discontinuities of the first kind, situated on the characteristics $x \pm at = \text{const.}$, i.e. we have the case of hyperbolic solutions with non-removable discontinuities. In the case of a finite rectangle $(0 \leqslant x \leqslant L, 0 \leqslant t \leqslant T)$ there will be a finite number of discontinuous characteristics.

We will prove that boundary-value problem (1) is conservative. In order to eliminate the atmospheric pressure P_0 from the system of external loads and not to have to take into account its work when the boundaries are displaced, and also in order to be able to ignore the initial elastic potential energy of the liquid arising from the compression by the atmospheric pressure P_0, we will take as the undeformed state of the liquid its state under normal conditions, i.e. when it is compressed by atmospheric pressure. The justification of this approach can be proved rigorously.

It can be shown that the integral of the total energy, apart from a constant factor, for problem (1) is

$$E(t) = b\psi^2(L, t) + \int_0^L (a^{-2}\psi_t^2 + \psi_x^2)dx \tag{5}$$

The following assertions hold.

Assertion 1. If the function $f(t)$ satisfies requirements 1° – 3° above, then $E(t) \in C$.

Assertion 2. The condition of compatibility at all points where no two (left and right) discontinuous characteristics cross is satisfied along the whole of the discontinuous characteristic, namely, the quantity $\psi_t \pm a\psi_x$ will vary continuously on passing through the corresponding characteristic of the discontinuity $x \mp at = \text{const}$.

Assertion 3. For any $t > \tau$ for which there is not a single crossing of the discontinuous left and right characteristics, we will have $dE/dt = 0$.

To prove the last assertion it is necessary to take into account the discontinuity of the integrand and to split integral (5) into parts with variable limits, which move with velocity $\pm a$ along the x axis.

Moreover, it was recalled above that over the whole finite rectangle $(0 \leqslant x \leqslant L, 0 \leqslant t \leqslant T)$ there will be a finite number of points where the right and left discontinuous characteristics cross.

Hence, when $\tau < t < T$ we have $E(t) \in C$ everywhere and $dE/dt = 0$ everywhere, with the exception, possibly, of a finite number of isolated points. Hence it follows that $E(t) = \text{const}$ when $t > \tau$, i.e. the following theorem is proved.

Theorem 1. Problem (1) is conservative if the external impulse $f(t)$ satisfies requirements 1–3.

We will consider further the limiting cases $b \to 0$ and $b \to \infty$. The case $b \to 0$ corresponds to an unlimited increase in the stiffness of the wall $x = L$, and $b = 0$ corresponds to a fixed boundary. It can be shown that as $b \to 0$ the solution, together with its first derivatives, tends uniformly with respect to x and t to the corresponding limiting values, which agree completely with those obtained when $b = 0$. This case is quite simple and will not be considered further.

Another situation is obtained when considering large b (ultrathin films). The solution in this case also will properly converge, uniformly with respect to both variables, to the solution when $b = \infty$ (a free surface). Also the first derivatives of function (3) will have anomalies, which we will consider in more detail later.

Consider the instant of time t which satisfies the condition $\tau + L/a < t < 2L/a$.

During this time the pulse can be completely reflected from the boundary $x = L$ and the reflected wave is unable to reach the boundary $x = 0$. By relations (2)–(4), the wave pattern of the pressure distribution transverse to the liquid layer will have the form

$$P(x) = \begin{cases} 0, & 0 \leqslant x < 2L - at \\ f\left(\frac{z}{a}\right) - 2be^{-bz}\int\limits_0^z f\left(\frac{z'}{a}\right)e^{bz'}dz', & 2L - at \leqslant x \leqslant L \end{cases} \tag{6}$$

$$z = x + at - 2L \geqslant 0$$

Suppose τ_i $(i = 1, 2, \dots, l)$ are the time coordinates of points of discontinuity of the pulse $f(t)$. Consider the situation after the i-th discontinuity in formula (6). The lower (non-zero) part of Eq. (6) when $b \gg 1$ gives

$$P_{a1}(u) = f\left(\tau_i + \frac{u}{a}\right) - 2e^{-bu}\left[f(\tau_i - 0) + b\int\limits_0^u f\left(\tau_i + \frac{u'}{a}\right)e^{bu'}du'\right] \tag{7}$$

$$u = z - a\tau_i = x + at - 2L - a\tau_i \geqslant 0$$

Carrying out some identical transformations on (7) and using properties 1°–3° of the specified load pulse, we can prove the following assertions.

Lemma 1. As $b \to \infty$, for any $\varepsilon > 0$ and

$$\varepsilon < u \leqslant a(\tau_{i+1} - \tau_i) \tag{8}$$

the following holds:

1. $P_{a1}(u) \to -f(\tau_i + u/a)$ uniformly in u;
2. the quantity $|dP_{a1}/du|$ is bounded by one and the same number, independent of ε.

Lemma 2. The quantity $|P_{a1}|$ is always bounded.

Hence, after the first reflection from the right boundary, asymptotic form (7) will hold. Further, the wave (7) reaches the left, free surface (property 1 of the pulse) and is reflected from it in the usual way, i.e. the value and direction of the motion are reversed. The tail behind the wave (7) and the velocity of the right boundary when $t > \tau + L/a$ will be infinitesimal and hence make an infinitesimal contribution, in view of the well-known property that the hyperbolic problem is well posed [2, 4].

To consider the wave pattern after the second reflection from the right boundary in the asymptotic form as $b \to \infty$ we can replace $f(\tau_i + u/a)$ by $[-P_{a1}(u)]$ in expression (7), while $f(\tau_i - 0)$, by virtue of the first assertion of Lemma 1, may not change. Hence, we obtain $P_{a2}(u)$ for

$$\tau + 3L/a < t < 4L/a, \quad u = x + at - a\tau_i - 4L \geqslant 0.$$

We can also prove Lemmas 1 and 2 for $P_{a2}(u)$. Here it is important to use the fact that these lemmas have already been proved for $P_{a1}(u)$. Extending the solution of the problem further in a similar way we can construct the following recurrence procedure

$$P_{a0}(u) = -f(\tau_i + u/a)$$

$$\tau + (2n-1)L/a < t < 2nL/a$$

$$u = x + at - a\tau_i - 2nL \geqslant 0, \quad n = 1, 2, \dots$$

$$P_{an}(u) = -P_{a,n-1}(u) - 2e^{-bu}\left[f(\tau_i - 0) - b\int\limits_0^u P_{a,n-1}(u')e^{bu}du'\right] \tag{9}$$

The following lemma and theorems hold for this, as indicated above.

Lemma 3. As $b \to \infty$, for any $\varepsilon \to \infty$ and when inequality (8) is satisfied, the quantity $|dP_{an}/du|$ is bounded by one and the same number, independent of ε, for any fixed n.

Theorem 2. As $b \to \infty$, for any $\varepsilon > 0$ and when inequality (8) is satisfied, $P_{an} \to -f(\tau_i + u/a)$ (uniformly in u for any fixed n.

Theorem 3. The quantity $|P_{an}|$ is everywhere bounded for any fixed n.

Theorem 4. If $f(\tau_i - 0) = f(\tau_i + 0)$, then for any fixed n as $b \to \infty$ we will have $P_{an}(u) \to -f(\tau_i + u/a)$ uniformly in u in any closed neighbourhood of $u = 0$ which does not contain neighbouring points of discontinuity.

We will consider the zone of anomalies in more detail. According to the theorems proved above, as $b \to \infty$ it will always be localized in a small right-sided ε-neighbourhood behind the discontinuous front of the reflected wave for any fixed reflection n from the right boundry.

After introducing the notation.

$$\xi = bu, \quad P_n(\xi) = P_{an}(\xi / b)$$

(ξ is a dimensionless quantity) the form of recurrence procedure (9) is somewhat simplified. It would have been considerably simpler if at the first step we had simply assumed $\tilde{P}_0(\xi) = -f(\tau_i + 0) = \text{const}$. In this connection we prove one more theorem.

Theorem 5. For any N and $\varepsilon > 0$ we have B_0 and $\delta > 0$ such that when $0 \leqslant \xi \leqslant b\delta$ we will have $|\tilde{P}_n - P_n| < \varepsilon$) and when $b\delta < \xi \leqslant ab(\tau_{i+1} - \tau_i)$ we will have $|P_n(\xi) + f|\tau_i + \xi/(ab)] < \varepsilon$, provided $n \leqslant N, b \geqslant B_0$ and the specified pulse $f(t)$ satisfies requirements 1–3.

Here the tilde denotes quantities corresponding to the simplified recurrence procedure. The proof of Theorem 5 is quite lengthy, although it does not present any particular difficulties; we will therefore not give it here. Hence, the zone of anomalies as $b \to \infty$ can be described with any accuracy by the simplified recurrence formula. Outside this anomalous zone the pressure wave will differ infinitesimally from the initial pulse with the opposite sign.

The simplified recurrence procedure can be reduced to the following simple form by the method of induction

$$\begin{aligned} &\tilde{P}_n(\xi) = -f^+ + (f^+ - f^-)A_n(\xi) \\ &A_n(\xi) = 2e^{-\xi}q_n(\xi), \quad f^\pm = f(\tau_i \pm 0) \\ &q_1(\xi) = 1, \quad q_n(\xi) = 1 - q_{n-1}(\xi) + 2\int_0^\xi q_{n-1}(\xi')d\xi' \end{aligned} \tag{10}$$

It can be seen that $q_n(\xi)$ are polynomials of degree $n - 1$. It should be noted that formulae (10) lose their meaning as $n \to \infty$; nevertheless, they told for any fixed n.

In solutions for $b = \infty$ (a free surface) all A_n must be assumed to be identical zeros. At the same time, for large b (ultrathin coatings) the solutions will have the form (10), i.e. they will consist of a normal component $(-f^+)$ and an anomalous component, where the anomalous component will vary considerably depending on the reflection number n.

The first four anomalies are shown in Fig. 1. Each odd anomaly will give a spike, two discontinuities in height exactly at the beginning of the front of the reflected wave. Each even anomaly begins from zero. All the anomalies, beginning with the second, have intermediate maxima, fairly substantial in value – almost one and a half discontinuities. All the anomalies differ considerably from one another and, naturally, have the abscissa axis as their horizontal asymptote. According to Theorem 4 all these anomalies, for any fixed n, disappear in the case of a continuous pulse and a fairly thin film.

In Fig. 2 we show wave patterns of the pressure for a unit step pulse of length $L/(3a)$ at the instants of time $2L/(3a)$ (a), $8L/(3a)$ (b), $14L/(3a)$ (c) and $20L/(3a)$ (d) respectively. The anomalies described above can be clearly seen on all the wave patterns, beginning with the second, behind both the front and rear discontinuous fronts of the reflected waves.

The mathematical nature of these phenomena consists of a peculiar defect in the convergence of the non-classical solutions of boundary-value problem (1). Waves with non-removable discontinuities, strictly speaking, belong to the class of generalized solutions of the equations of mathematical physics, for which

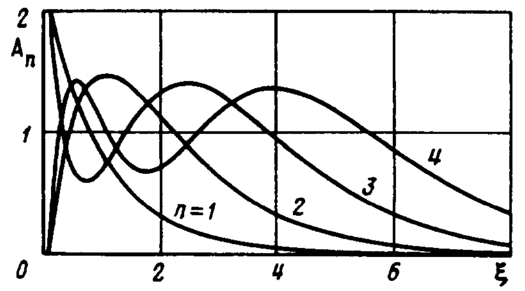

Fig. 1

the convergence is much weaker than for classical solutions [4]. In this case also, one observes non-uniform convergence of the derivatives of the uniformly converging solution.

The physical nature of anomalies (10) consists of the fact that model (1) enables the velocity to change abruptly on the free surface $b = \infty$ and to change only continuously for any finite b (i.e. for any film as thin and as light as desired).

Note that for incompressible obstacles, along which the perturbation propagates with a finite velocity, system (1) will be exact rather than approximate, since such obstacles can be regarded as added masses, that in turn, lead to a mixed boundary condition in (1). For such films, anomalies (10) will, in fact, occur. If the material of the film is compressible, then, in the exact formulation, it is also necessary to take wave processes along the film thickness into account. In this case anomalies (10) may attenuate as one approaches the acoustic impedances of the carrier layer (a liquid in the case considered) and of the obstacle. System (1) does not describe these effects.

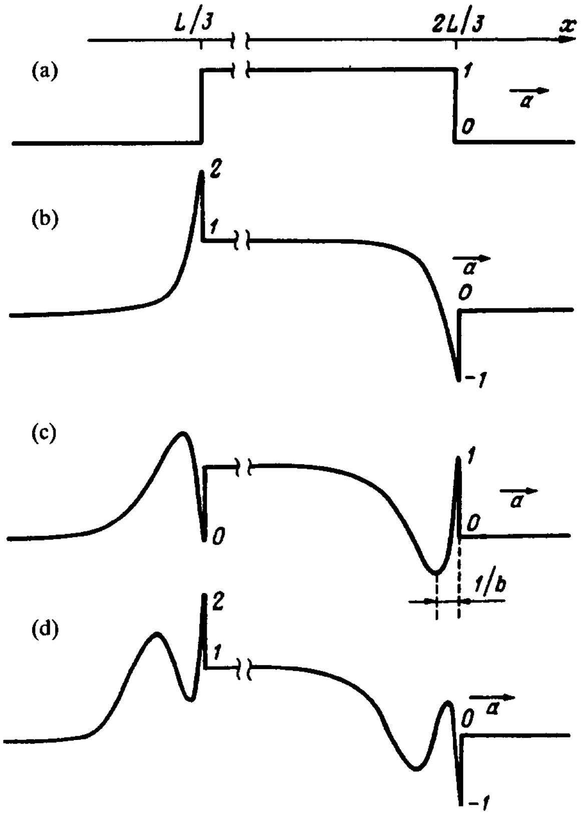

Fig. 2

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